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R. Klages, G. Radons, I.M. Sokolov (eds.): Anomalous Transport: Foundations and Applications

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This tome, a challenge to the weight lifter, and undertaken as a tribute to the late Radu Balescu, consists of nineteen articles by experts on various approaches to the formulation and solution of problems suggested by the mathematical and physical aspects of anomalous, i.e., non-diffusive transport. These range from the realm of pure mathematics to experiments, but heavily weighted towards the former. I'm not familiar with all of the topics discussed, and learned a lot, if not all of the subtleties inherent in them, as I'm sure will the reader. Only a few of the articles can be mentioned in a finite review which might hopefully serve to provide as a useful soupcon of the subject.

A first article that struck my attention is one on fractional derivatives by R. Hilfer which pointed out such derivatives were first defined by Leibniz, but also that the theory has been considerably enlarged by an army of distinguished mathematicians up to the present era. This plethora of possible definitions raises the question of what might be the physical consequences of analysis based on fractional diffusion equations in which the definition of fractional derivative differs from the almost standard Liouville-Riemann definition that appears in all analyses. This general question, as far as I'm aware, has neither been raised nor considered in any detail in the literature of statistical physics. One obvious advantage of dealing with fractional diffusion equations is that solving them often involves the same techniques as solving standard diffusion equations, except that the exponential functions of time that appear in eigenfunction expansions using the Liouville-Riemann definition are generally replaced by Mittag-Leffler functions.

In contrast to this approach, possibly the most popular phenomenological model in current use in an extensive variety of applications in the physical sciences and in finance is the continuous-time random walk (CTRW), originally suggested in 1965 [1]. Early applications include applications to transport in amorphous solids [2], and to transport in amorphous semiconductors [3]. Various peripheral details related to the CTRW are described in several

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articles in the book, e.g., R. Gorenflo and F. Mainardi describe properties of the Mittag-Leffler function which arise both in the use of fractional diffusion equations and the CTRW. It should be emphasized that neither of the general approaches mentioned reaches the level of fundamental physics, so that it is impossible to fully analyze their strengths and weaknesses. The most comprehensive efforts along these lines has been done in the case of the CTRW by the group led by B. Berkowitz and H. Scher at the Weizmann Institute of Science. They have extensively applied CTRW formalism to a variety of phenomena in hydrology [4–6]. Contributions from this group are in both the theoretical and experimental areas, the latter of which confirm main points of the formulation in terms of the CTRW. Unfortunately this work is given only peripheral mention in the volume under review, whereas it has provided a major contribution to the field of hydrology. However, it should also should be noted that a formulation in terms of fractional derivatives can be found from the CTRW by retaining the lowest order terms in an expansion of the characteristic function corresponding to a pausing time density having the asymptotic form $t^{-(1+\alpha)}$, $0 < \alpha < 1$.

Closely related to the formulation of anomalous transport problems in terms of fractional derivatives are models based on Lévy flights, in which the probability density of steps of a random walk decay asymptotically with length, $|x|$, as $|x|^{-(1+\alpha)}$ with $0 < \alpha < 2$. Such flights have been observed in a number of situations in nature as cited by A. Chechkin, R. Metzler, J. Klafter and V. Yu. Gonchar in their very readable tutorial review of aspects of the theory. A related article by D. del-Castillo-Negrete discusses fractional diffusion models of anomalous transport, i.e., based on fractional derivatives. However, as mentioned, these seem to me to be always open to the objection that they are purely theoretical, and not tied to experimental data in the same sense as have CTRW-based models. Indeed, difficulties which manifest themselves as negative concentrations are mentioned in the article by S. Yuste, K. Lindenberg and J. Ruiz-Lorenzo and in [7].

E. Barkai has given a brief discussion of a property that I would call the occupation time of a stochastic process. Suppose one is given a trajectory of such a process for a time t and a specific volume V. The occupation time is the fraction of time spent by the trajectory in V during t in the limit $t \to \infty$. In the case of Brownian motion a well-known result is that this is given by the arcsine law [8]. Barkai's article focuses mainly on recent investigations of changes of the assumptions that imply the arcsine law, such as those which arise in the case of subdiffusive mean-squared displacements and when a force field acts on the particle. There is much of interest in this introductory article, which might also have benefited by a discussion of more physical applications rather than the more abstract notion of weak ergodicity breaking.

The second part of the monograph is devoted to what the editors term deterministic theories of anomalous transport. Perhaps the most familiar example of a member of this class is a system that evolves chaotically [9], i.e., the time-dependent behavior of the system can be calculated exactly, but whose qualitative behavior can be described as being stochastic. The background and some elementary results for possibly the most widely studied model in this class, that of Fermi, Pasta and Ulam [10], is summarized by S. Lepri, R. Livi and A. Politi.[1] In the FPU model rather crude simulations showed that nonlinear coupling between atoms in a one-dimensional lattice produced soliton-like behavior in the interchange of energy rather

[1] The calculations were made on the MANIAC1 at Los Alamos, which the authors cite as being the first programmable computer. However, the pride of place is generally given to the ENIAC at the Aberdeen Proving Grounds in Maryland which was in operation approximately ten years earlier. I was stationed there at its termination in 1954.

than the thermalization initially anticipated by the distinguished investigators. A rich collection of further problems related to heat conduction is reviewed in the remainder of the article which is a useful introduction to an area of physics with problems whose solutions still pose significant challenges.

A third section of the book is devoted to transport in disordered systems, e.g., glasses or porous materials such as occur in hydrology. The section is prefaced by a clear introductory article by J-P. Bouchaud which summarizes his and his collaborators' work over a period of approximately fifteen years on physical models leading to what I would term non-standard relaxation behavior. These are exemplified by models in which lattice random walks may be restrained by randomly located trapping lattice points for random amounts of time. This, in turn, leads to quite intricate relaxation behavior characterized by multiple phases, as is observed in a number of disordered systems.

The final section gets down to what I consider the meat of anomalous transport; that is; applications to complex systems and experimental results. A prominent theme in this class of investigations is whether the language of physics can provide any insights into what are primarily problems of curve fitting in non-physics contexts. The surprising answer to this that indeed it can, at least in some interesting cases. One of these is discussed in an article by D. Brockmann on the development of models for statistical properties of human travel based on data on the geographic circulation of bank notes. A study of a considerable amount of such data indicated that the dispersal of bank notes, initially at Omaha, Nebraska was quite satisfactorily described in terms of a Lévy flight except at very short distances. Brockmann concludes his discussion by relating the pdf for having traveled the scalar distance r in time t, $W(r,t)$, derived from a CTRW, provided that the pdf's of a single displacement and a single interjump time have the asymptotic forms $p(\mathbf{r}) \sim r^{-(2+\beta)}$, $0 < \beta < 2$ and $\psi(t) \sim t^{-(1+\alpha)}$, $0 < \alpha < 1$.

In summary, this collection of articles will give the reader a feeling for some, but hardly all, of the approaches currently being used to model anomalous transport. Only a few articles have been mentioned, which is not to belittle the remainder which all contain at least some information of interest, but possibly with greater technical detail than those mentioned.

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